Estimation and Inference of Asymmetric Impulse Response Functions

Preliminary version

Narek Ohanyan*

American University of Armenia

October 20, 2024

Abstract

In this paper, I study asymmetric moving average processes and provide theory for such processes, their properties and conditions for convergence, stationarity, etc. I also propose simple 2-step least squares procedure for estimating asymmetric impulse response functions by local projections. Then, I generalize the estimation method to vector asymmetric moving average processes and show that asymmetric impulse response functions may be estimated using the proposed methodology. Finally, Monte Carlo analysis validates the proposed method with satisfactory performance both in small and large samples.

Keywords: asymmetric moving average, asymmetric effects, impulse response functions, local projections

JEL Classification Codes: C13, C22, C32

*Email: narek.ohanyan@aua.am.

Introduction

Many popular estimation methods, e.g. Vector Autoregressions and Local Projections, assume that the data generating process is symmetric and the effects of positive and negative innovations are identical. However, in many cases, asymmetric effects of innovations may be present. For example, monetary policy shocks are often considered to have different macroeconomic effects. On the other hand, price and wage rigidities may lead to asymmetric effects of positive and negative economic shocks. Finally, the irreversibity of investment decisions may lead to asymmetric effects of positive and negative productivity shocks.

In all these cases, it is important to estimate the asymmetric effects of innovations on the variable of interest. In this paper, I extend the concept of moving average processes to more general asymmetric moving average processes and provide theory for such processes and their properties. I also propose a simple 2-step estimation procedure based on local projections regressions. Different, from other methods, the proposed method (1) is simple to implement (2) allows for non-parametric estimation of impulse response functions, (3) allows for testing for asymmetric effects by a simple Wald test. I study the performance of the proposed estimator by Monte Carlo analysis and find that it has satisfactory performance both in small and large samples.

Alternative estimation methods attempt to allow for non-linear or asymmetric effects but have several limitations. Wecker (1981) and Brånnås and De Gooijer (1994) propose Maximum Likelihood estimation for the asymmetric effects in moving average models. However, this method is often challenging due to its high computational costs as it may require estimation of large number of parameters. For instance, to estimate the asymmetric effects of a variable over a horizon of 36 periods (common with monthly data), one would need to estimate 72 parameters.

On the other hand, Barnichon and Matthes (2018) propose a method of approximating impulse responses by Gaussian function approximation. This procedure allows for asymmetric effects but may require large number of basis functions for satisfactory approximation. Moreover the estimation requires use of Maximum Likelihood or Bayesian methods, which have significantly higher computational costs.

Finally, Jordà (2005) proposes "flexible" local projections trying to allow for non-linear effects for projecting the variable of interest onto polynomials (quadratic, cubic, etc.) of the dependent variable. However, this method (1) doesn't allow for asymmetric effects of innovations, (2) may require higher order polynomials for satisfactory approximation of the true data generating process.

Asymmetric time series processes have been studied in the literature, though not extensively. Wecker (1981) proposes an asymmetric moving average ($asMA$) process, where the moving average coefficients are different for positive and negative innovations. Later, Brånnås and

De Gooijer (1994) extend the concept of asymmetric moving average processes to Autoregressive Asymmetric Moving Average ($ASasMA$) processes with symmetric autoregressive terms and asymmetric moving average terms. On the other hand, Brännäs and Gooijer (2004) introduced the concept of $asMA - asQGARCH$ models allowing to asymmetric effects on the conditional volatility of the process. Furthermore, De Gooijer (2021) and Brännäs et al. (2012) extended the concept of asymmetric moving average processes to multivariate case, proposing Asymmetric Vector Moving Average ($asVMA$) processes, proposing Maximum Likelihood estimation for such processes.

The rest of the paper is organized as follows. Sections 1 and 2 provide theory for univariate and multivariate asymmetric moving average processes and their properties. Section 3 proposes the estimation procedure for asymmetric moving average processes and provides a simple Wald test for asymmetric effects. Section 4 extends the proposed estimation methodology to multivariate processes and asymmetric impulse response functions. Finally, Section 5 investigates the performance of the proposed estimators by Monte Carlo analysis.

1 Asymmetric Moving Average Processes

I extend the concept of moving average processes to more general asymmetric moving average processes, which I refer to as $asMA(\cdot)$ processes, in line with the language of Wecker (1981).

Definition 1.1 *An Asymmetric Moving Average $asMA(\infty)$ process y_t is given by*

$$y_t = c + \theta_0(\varepsilon_t) + \theta_1(\varepsilon_{t-1}) + \theta_2(\varepsilon_{t-2}) + \dots$$

where ε_t is independent and identically distributed random variable with symmetric probability density function $f(\cdot)$, such that

$$\varepsilon_t \sim iid(0, \sigma^2_\varepsilon) \qquad f_\varepsilon(\varepsilon_t) = f_\varepsilon(-\varepsilon_t)$$

and $\theta_i(\cdot), i \geq 0$ are asymmetric linear functions, given by

$$\theta_i(\varepsilon_{t-i}) = \theta_i^+ \varepsilon_{t-i}^+ + \theta_i^- \varepsilon_{t-i}^- \qquad\qquad for \quad i \geq 0$$

where ε_t^+ and ε_t^- are positive and negative realizations of ε_t, θ_i^+, θ_i^- are the corresponding coefficients and $\theta_0^+ = \theta_0^- = 1$.

Remark *Note that if y_t is an $asMA(\infty)$ process such that $\theta_i^+ = \theta_i^-, \forall i$ then y_t reduces to a linear and symmetric $MA(\infty)$ process.*

Next, I characterize the properties of the asymmetric moving average processes. In particular, I derive the mean, variance and autocovariance functions of the process y_t in Theorem 1.1.

Theorem 1.1 *Let y_t be an infinite Asymmetric Moving Average $asMA(\infty)$ process of form*

$$y_t = c + \theta_0(\varepsilon_t) + \theta_1(\varepsilon_{t-1}) + \theta_2(\varepsilon_{t-2}) + \dots \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$$

then

$$a) \quad \mathrm{E}[y_t] = c + \frac{1}{2}\left(\sum_{i=0}^{\infty} \theta_i^+ - \sum_{i=0}^{\infty} \theta_i^-\right) \mu_\varepsilon^+$$

$$b) \quad \mathrm{Var}[y_t] = \frac{1}{2}\left(\sum_{i=0}^{\infty} {\theta_i^+}^2 + \sum_{i=0}^{\infty} {\theta_i^-}^2\right) \sigma_\varepsilon^2 - \frac{1}{4}\sum_{i=0}^{\infty}\left(\theta_i^+ - \theta_i^-\right)^2 {\mu_\varepsilon^+}^2$$

$$c) \quad \mathrm{Cov}[y_t, y_{t-s}] = \frac{1}{2}\left(\sum_{i=0}^{\infty} \theta_{s+i}^+ \theta_i^+ + \sum_{i=0}^{\infty} \theta_{s+i}^- \theta_i^-\right) \sigma_\varepsilon^2 - \frac{1}{4}\sum_{i=0}^{\infty}\left(\theta_{s+i}^+ - \theta_{s+i}^-\right)\left(\theta_i^+ - \theta_i^-\right) {\mu_\varepsilon^+}^2$$

where $\mu_\varepsilon^+ \equiv \mathrm{E}[\varepsilon_t \,|\, \varepsilon_t \geq 0]$.

We can see from Theorem 1.1 that, different to symmetric moving average processes, the mean of the process y_t is not necessarily equal to the constant term c, but depends on the asymmetry of the moving average coefficients. Also, the mean of the process y_t is a function of the conditional mean of the positive (as well as negative) innovations μ_ε^+.[1]

Similarly, the variance and autocovariance functions of the process y_t are functions of the asymmetry of the moving average coefficients and the conditional mean of the positive innovations μ_ε^+.

To better understand the properties of the asymmetric time series, suppose we have an $asMA(\infty)$ process with following parameters

$$\theta_i^+ = \phi_+^i \qquad \theta_i^- = \phi_-^i \qquad \varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$$

that is, the process is a mix of two $AR(1)$ processes with parameters ϕ_+ and ϕ_- and Gaussian innovations.

[1] For innovations ε_t with symmetric probability density function $f(\cdot)$, the conditional mean of the positive innovations satisfies $\mu_\varepsilon^+ = -\mu_\varepsilon^-$.

Then, $\mu_\varepsilon^+ = \sqrt{2/\pi}\sigma_\varepsilon$ and the mean, variance and autocovariance functions of y_t are

$$\mathrm{E}[y_t] = c + \left(\frac{1}{1-\phi_+} - \frac{1}{1-\phi_-}\right)\frac{\sigma_\varepsilon}{\sqrt{2\pi}}$$

$$\mathrm{Var}[y_t] = \left(\frac{1}{1-\phi_+^2} + \frac{1}{1-\phi_-^2}\right)\frac{\sigma_\varepsilon^2}{2} - \left(\frac{1}{1-\phi_+^2} - \frac{2}{1-\phi_+\phi_-} + \frac{1}{1-\phi_-^2}\right)\frac{\sigma_\varepsilon^2}{2\pi}$$

$$\mathrm{Cov}[y_t, y_{t-s}] = \left(\frac{\phi_+^s}{1-\phi_+^2} + \frac{\phi_-^s}{1-\phi_-^2}\right)\frac{\sigma_\varepsilon^2}{2} - \left(\frac{\phi_+^s}{1-\phi_+^2} - \frac{\phi_+^s}{1-\phi_+\phi_-} - \frac{\phi_-^s}{1-\phi_+\phi_-} + \frac{\phi_+^s}{1-\phi_-^2}\right)\frac{\sigma_\varepsilon^2}{2\pi}$$

Fig. 1 plots the autocorrelation functions of an $asMA(\infty)$ process consisting of two $AR(1)$ processes with parameters ϕ_+ and ϕ_- for positive and negative innovations, respectively. In the left panel, the process is given by $\phi_+ = 0.8$ and $\phi_- = 0.4$, while in the right panel, the persistence of the positive and negative processes is significantly different, with $\phi_+ = 0.9$ and $\phi_- = 0.2$.

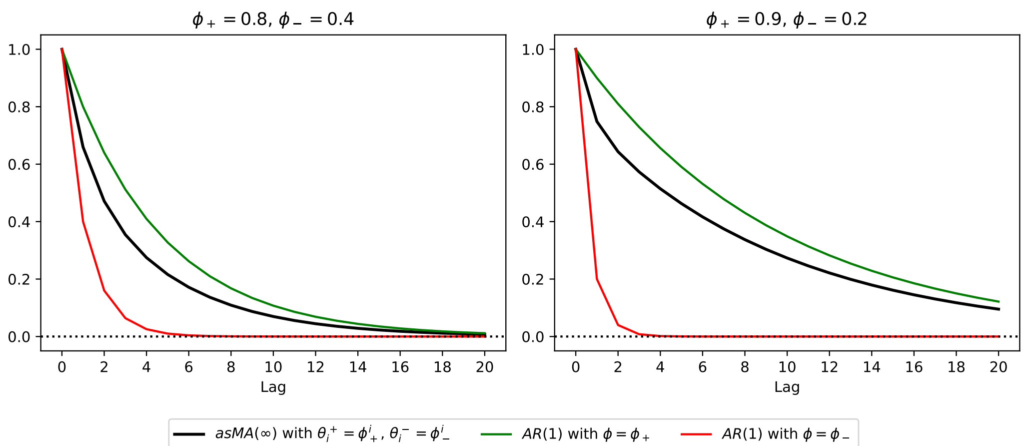


Figure 1: Autocorrelation functions of $asMA(\infty)$ process with different parameters

The figure shows that the ACF of the $asMA(\infty)$ process is a mix of the ACFs of the two $AR(1)$ processes, but not necessarily a linear combination of them. Instead, it is mainly determined by the largest (in absolute value) of the two persistence parameters ϕ_+ and ϕ_-.

The similarity of the ACF of the $asMA(\infty)$ process to the ACF of the $AR(1)$ process implies that although the $AR(1)$ process may be a good approximation for the $asMA(\infty)$ process, the $AR(1)$ process fails short in capturing the asymmetry of the process.

1.1 Representation

In this section, I provide a representation of the asymmetric moving average processes in terms of its lags and a white noise process. This representation is useful for understanding the properties of the asymmetric time series. It also becomes handy for estimation of the parameters of the asymmetric moving average processes.

Theorem 1.2 *Let y_t be an infinite Asymmetric Moving Average $asMA(\infty)$ process of form*

$$y_t = c + \theta_0(\varepsilon_t) + \theta_1(\varepsilon_{t-1}) + \theta_2(\varepsilon_{t-2}) + \dots \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$$

Then, y_t can be expressed as

$$y_t = b + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \eta_t \qquad \eta_t \sim WN(0, \sigma_\eta^2)$$

where $\eta = \varepsilon + \xi$, with $\xi \sim WN(0, \sigma_\xi^2)$, such that

$$\mathrm{E}[\xi_t] = 0 \qquad \mathrm{E}[\xi_t \xi_{t-i}] = 0 \qquad \mathrm{E}[\xi_t \varepsilon_t] = 0 \qquad \mathrm{E}[\xi_t y_{t-j}] = 0$$

for $i \geq 1$ and $j \geq 1$.

2 Asymmetric Vector Moving Average Processes

Definition 2.1 *An Asymmetric Vector Moving Average $asVMA(\infty)$ process $\boldsymbol{y}_t$ is given by*

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Theta}_0(\boldsymbol{\varepsilon}_t) + \boldsymbol{\Theta}_1(\boldsymbol{\varepsilon}_{t-1}) + \boldsymbol{\Theta}_2(\boldsymbol{\varepsilon}_{t-2}) + \dots$$

where $\boldsymbol{\varepsilon}_t = [\varepsilon_{1,t}, \dots, \varepsilon_{n,t}]'$ is a vector of independent and identically distributed random variables with symmetric probability density functions $f_{\varepsilon_k}(\cdot)$, such that

$$\boldsymbol{\varepsilon}_t \sim iid(\boldsymbol{0}, \boldsymbol{\Sigma}_\varepsilon) \qquad \boldsymbol{\Sigma}_\varepsilon = diag(\sigma_1^2, \dots, \sigma_n^2) \qquad f_{\varepsilon_k}(\varepsilon_{k,t}) = f_{\varepsilon_k}(-\varepsilon_{k,t})$$

and $\boldsymbol{\Theta}_i(\cdot), i \geq 0$ are asymmetric vector valued functions, given by

$$\boldsymbol{\Theta}_i(\boldsymbol{\varepsilon}_{t-i}) = \boldsymbol{\Theta}_i^+ \boldsymbol{\varepsilon}_{t-i}^+ + \boldsymbol{\Theta}_i^- \boldsymbol{\varepsilon}_{t-i}^- \qquad \text{for} \quad i \geq 0$$

where $\boldsymbol{\varepsilon}_t^+$ and $\boldsymbol{\varepsilon}_t^-$ are elements-wise positive and negative realizations of $\boldsymbol{\varepsilon}_t$, $\boldsymbol{\Theta}_i^+$, $\boldsymbol{\Theta}_i^-$ are $n \times n$ corresponding coefficient matrices and $\boldsymbol{\Theta}_0 \equiv \boldsymbol{\Theta}_0^+ = \boldsymbol{\Theta}_0^-$ with normalization of its diagonal elements to 1.

Remark *Note that if $\boldsymbol{y}_t$ is an $asVMA(\infty)$ process such that $\boldsymbol{\Theta}_i^+ = \boldsymbol{\Theta}_i^-, \forall i$ then $\boldsymbol{y}_t$ reduces to a linear and symmetric $VMA(\infty)$ process.*

Next, in Theorem 2.1, I derive expressions for the mean, variance and autocovariance functions of the process $\boldsymbol{y}_t$.

Theorem 2.1 *Let $\boldsymbol{y}_t$ be an infinite Asymmetric Vector Moving Average $asVMA(\infty)$ process of form*

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Theta}_0(\boldsymbol{\varepsilon}_t) + \boldsymbol{\Theta}_1(\boldsymbol{\varepsilon}_{t-1}) + \boldsymbol{\Theta}_2(\boldsymbol{\varepsilon}_{t-2}) + \ldots \qquad \boldsymbol{\varepsilon}_t \sim iid(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

then

a) $\mathrm{E}[\boldsymbol{y}_t] = \boldsymbol{c} + \frac{1}{2}\left(\sum_{i=0}^{\infty} \boldsymbol{\Theta}_i^+ - \sum_{i=0}^{\infty} \boldsymbol{\Theta}_i^-\right) \boldsymbol{\mu}_\varepsilon^+$

b) $\mathrm{Var}[\boldsymbol{y}_t] = \frac{1}{2}\sum_{i=0}^{\infty}\left(\boldsymbol{\Theta}_i^+ \boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Theta}_i^{+'} + \boldsymbol{\Theta}_i^- \boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Theta}_i^{-'}\right) - \frac{1}{4}\sum_{i=0}^{\infty}\left(\boldsymbol{\Theta}_i^+ - \boldsymbol{\Theta}_i^-\right) \boldsymbol{\mu}_\varepsilon^+ \boldsymbol{\mu}_\varepsilon^{+'} \left(\boldsymbol{\Theta}_i^{+'} - \boldsymbol{\Theta}_i^{-'}\right)$

c) $\mathrm{Cov}[\boldsymbol{y}_t, \boldsymbol{y}_{t-s}] = \frac{1}{2}\sum_{i=0}^{\infty}\left(\boldsymbol{\Theta}_{s+i}^+ \boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Theta}_i^{+'} + \boldsymbol{\Theta}_{s+i}^- \boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Theta}_i^{-'}\right) - \frac{1}{4}\sum_{i=0}^{\infty}\left(\boldsymbol{\Theta}_{s+i}^+ - \boldsymbol{\Theta}_{s+i}^-\right) \boldsymbol{\mu}_\varepsilon^+ \boldsymbol{\mu}_\varepsilon^{+'} \left(\boldsymbol{\Theta}_i^{+'} - \boldsymbol{\Theta}_i^{-'}\right)$

where $\boldsymbol{\mu}_\varepsilon^+ \equiv \mathrm{E}[\boldsymbol{\varepsilon}_t \,|\, \boldsymbol{\varepsilon}_t \geq 0]$.

2.1 Representation

Theorem 2.2 *Let $\boldsymbol{y}_t$ be an infinite Asymmetric Vector Moving Average $asVMA(\infty)$ process of form*

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Theta}_0(\boldsymbol{\varepsilon}_t) + \boldsymbol{\Theta}_1(\boldsymbol{\varepsilon}_{t-1}) + \boldsymbol{\Theta}_2(\boldsymbol{\varepsilon}_{t-2}) + \ldots \qquad \boldsymbol{\varepsilon}_t \sim iid(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

Then, $\boldsymbol{y}_t$ can be expressed as

$$\boldsymbol{y}_t = \boldsymbol{b} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2 \boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\eta}_t \qquad \boldsymbol{\eta}_t \sim WN(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

where $\boldsymbol{\eta} = \boldsymbol{\varepsilon} + \boldsymbol{\xi}$, with $\boldsymbol{\xi} \sim WN(\mathbf{0}, \boldsymbol{\Sigma}_\xi)$, such that

$$\mathrm{E}[\boldsymbol{\xi}_t] = \mathbf{0} \qquad \mathrm{E}[\boldsymbol{\xi}_t \boldsymbol{\xi}_{t-i}'] = \mathbf{0} \qquad \mathrm{E}[\boldsymbol{\xi}_t \boldsymbol{\varepsilon}_t'] = \mathbf{0} \qquad \mathrm{E}[\boldsymbol{\xi}_t \boldsymbol{y}_{t-j}'] = \mathbf{0}$$

for $i \geq 1$ and $j \geq 1$.

3 Estimation of asMA Processes

In Section 1, I showed that any $asMA$ process has a Wold representation with innovations η_t that consist of the structural shocks ε_t and ξ_t, which is a function of past structural shocks. Next, I derive a representation of $asMA(\infty)$ processes that can be used to estimate the parameters of the process.

Theorem 3.1 *Let y_t be an infinite Asymmetric Moving Average $asMA(\infty)$ process of form*

$$y_t = c + \theta_0(\varepsilon_t) + \theta_1(\varepsilon_{t-1}) + \theta_2(\varepsilon_{t-2}) + \dots \qquad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$$

Then, y_{t+h} can be expressed as

$$y_{t+h} = \beta_{0h} + \beta_{1h} y_{t-1} + \beta_{2h} y_{t-2} + \dots + \theta_h^+ \varepsilon_t^+ + \theta_h^- \varepsilon_t^- + v_{t+h}$$

with

$$\mathrm{E}[v_{t+h}] = 0 \qquad \mathrm{E}[\varepsilon_t^+ v_{t+h}] = 0 \qquad \mathrm{E}[\varepsilon_t^- v_{t+h}] = 0 \qquad \mathbb{E}[y_{t-j} v_{t+h}] = 0$$

for all $h \geq 1$ and $j \geq 1$.

Therefore, the equation above can be estimated by the Least Squares method if there were proxies for the error terms ε_t^+ and ε_t^-.

Theorem 3.1 shows that the h-period ahead forecast of an $asMA(\infty)$ process can be written as a function of its current and past values, the structural shocks ε_t and, and the forecast errors v_{t+h}. Hence, the coefficients of an $asMA(\infty)$ process can be estimated by the Least Squares method if there were proxies for ε_t^+ and ε_t^-.

As proxies for the structural shocks ε_t, I propose using the residuals of the Wold representation of an $asMA(\infty)$ process $\widehat{\eta}_t$.

Hence, consider 2-step estimation scheme, as

$$\begin{aligned} y_t &= \boldsymbol{x}_t' \boldsymbol{\beta}_1 + \eta_t \\ y_{t+h} &= \boldsymbol{x}_t' \boldsymbol{\beta}_{2h} + \theta_h^+ \widehat{\eta}_t^+ + \theta_h^- \widehat{\eta}_t^- + v_{t+h} \end{aligned}$$

where $\boldsymbol{x}_t = [1, y_{t-1}, \dots, y_{t-p}]'$.

Therefore, the residual of the first-stage regression $\widehat{\eta}_t$ is a proxy of ε_t that also includes ξ_t as random noise.

The estimation problem poses three challenges. First, the Wold representation of an $asMA(\infty)$ process may be infinite and many lags of the dependent variable may be needed to approximate the Wold representation.

Second, the residuals of the first-stage regression $\widehat{\eta}_t$ are imperfect proxies for the structural shocks ε_t, since they also include ξ_t as random noise.

Third, the second stage regression includes the positive and negative residuals as different regressors. Correct inference requires that the signs of the residuals coincide with the signs of the structural shocks.

In the following sections, I test the performance of the proposed method in Monte Carlo analysis and find that it performs well in both finite and infinite samples and that the above-mentioned challenges have little effect on the estimation in practice.

3.1 A simple Wald test for asymmetric effects

Consider the second step of the estimation with

$$y_{t+h} = \boldsymbol{x}_t' \boldsymbol{\beta}_{2h} + \theta_h^+ \widehat{\eta_t^+} + \theta_h^- \widehat{\eta_t^-} + v_{t+h}$$

The equation above allows for a simple procedure to test for asymmetric effects. If there are no asymmetric effects, then we would have $\theta_h^+ = \theta_h^- = \theta_h$ and the corresponding estimators will satisfy $\widehat{\theta}_h^+ \xrightarrow{p} \theta_h$ and $\widehat{\theta}_h^- \xrightarrow{p} \theta_h$. However, if there are asymmetric effects, then $\widehat{\theta}_h^+ \xrightarrow{p} \theta_h^+$ and $\widehat{\theta}_h^- \xrightarrow{p} \theta_h^-$.

Therefore, the presence of asymmetric effects can be tested with a Wald test with the hypothesis $H_0 : \theta_h^+ = \theta_h^-$ and test statistic $\widehat{\theta}_h^+ - \widehat{\theta}_h^-$. The test can be carried out both for particular horizon h from the corresponding regression equation, or for all horizons with the hypothesis $H_0 : \theta_h^+ = \theta_h^-, \forall h$ by jointly estimating regressions for all horizons h.

4 Estimation of asVMA Processes

In Section 2, I showed that, similar to univariate processes, any $asVMA$ process has a Wold representation with innovations $\boldsymbol{\eta}_t$ that consist of the structural shocks $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\xi}_t$. Below, I derive a representation of $asVMA(\infty)$ processes that can be used to estimate the impulse response functions of the process.

Theorem 4.1 *Let $\boldsymbol{y}_t$ be an infinite Asymmetric Moving Average $asVMA(\infty)$ process of form*

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Theta}_0(\boldsymbol{\varepsilon}_t) + \boldsymbol{\Theta}_1(\boldsymbol{\varepsilon}_{t-1}) + \boldsymbol{\Theta}_2(\boldsymbol{\varepsilon}_{t-2}) + \dots \qquad \boldsymbol{\varepsilon}_t \sim iid(\boldsymbol{0}, \boldsymbol{\Sigma}_\varepsilon)$$

Then, $\boldsymbol{y}_{t+h}$ can be expressed as

$$\boldsymbol{y}_{t+h} = \boldsymbol{b}_{0h} + \boldsymbol{B}_{1h}\boldsymbol{y}_{t-1} + \boldsymbol{B}_{2h}\boldsymbol{y}_{t-2} + \dots + \boldsymbol{\Theta}_h^+ \boldsymbol{\varepsilon}_t^+ + \boldsymbol{\Theta}_h^- \boldsymbol{\varepsilon}_t^- + \boldsymbol{v}_{t+h}$$

with

$$\mathrm{E}[\boldsymbol{v}_{t+h}] = \boldsymbol{0} \qquad \mathrm{E}[\boldsymbol{\varepsilon}_t^+ \boldsymbol{v}_{t+h}'] = \boldsymbol{0} \qquad \mathrm{E}[\boldsymbol{\varepsilon}_t^- \boldsymbol{v}_{t+h}'] = \boldsymbol{0} \qquad \mathbb{E}[\boldsymbol{y}_{t-j} \boldsymbol{v}_{t+h}'] = \boldsymbol{0}$$

for all $h \geq 1$ and $j \geq 1$.

The dynamic effects of innovations can be estimated by the local projections regressions provided that we have proxies for the structural errors $\boldsymbol{\varepsilon}_t^+$ and $\boldsymbol{\varepsilon}_t^-$.

The proxies for the structural shocks $\boldsymbol{\varepsilon}_t$, may be obtained from the residuals of the Wold representation of an $asVMA(\infty)$ process $\widehat{\boldsymbol{\eta}}_t$ by common identification strategies in the literature, such as recursive ordering, sign restrictions, external instruments

Then using the estimates of the structural shocks, the impulse response functions of the $asVMA(\infty)$ process can be estimated by the following 2-step methodology

$$\begin{aligned}\boldsymbol{y}_t &= \boldsymbol{b} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\eta}_t \\ \boldsymbol{y}_{t+h} &= \boldsymbol{b}_{0h} + \boldsymbol{B}_{1h}\boldsymbol{y}_{t-1} + \boldsymbol{B}_{2h}\boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\Theta}_h^+\widehat{\boldsymbol{e}}_t^+ + \boldsymbol{\Theta}_h^-\widehat{\boldsymbol{e}}_t^- + \boldsymbol{v}_{t+h}\end{aligned}$$

where $\widehat{\boldsymbol{e}}_t^+$ and $\widehat{\boldsymbol{e}}_t^-$ are the estimates of the structural shocks $\boldsymbol{\varepsilon}_t^+$ and $\boldsymbol{\varepsilon}_t^-$ obtained from $\widehat{\boldsymbol{\eta}}_t$.

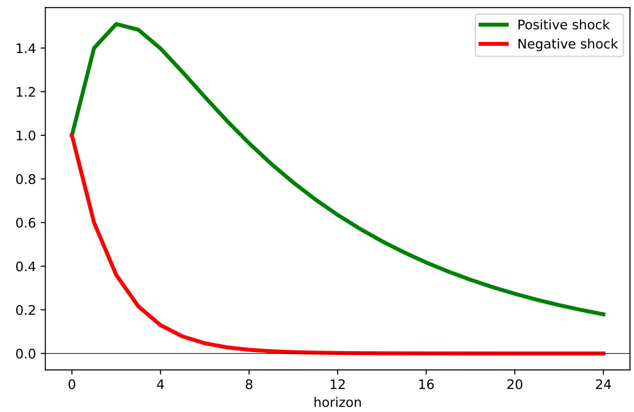
5 Monte Carlo analysis

5.1 Estimation of asMA processes

The previous sections developed theoretical background for asymmetric moving average processes and provided a simple method for estimating parameters of such processes via simple least squares regressions.

In this section I demonstrate the proposed methods in Monte Carlo analysis by first generating $asMA$ processes with known parameters and estimating them with the proposed methods.

First, I generate $asMA$ processes with following features. The parameters corresponding to positive innovations are given by the MA representation of an $AR(2)$ process with parameters $\phi_1^+ = 1.4$ and $\phi_2^+ = -0.45$. On the other hand, the parameters corresponding to negative innovations are given by the MA representation of a $AR(1)$ process with parameters $\phi_1^- = 0.6$. Consistent with the assumption of symmetric innovations, I draw innovations from a standard normal distribution. For the simulations, I consider sample sizes of 100, 200, 500, 1000 observations and generate 1000 series of $asMA$ processes for each sample size.

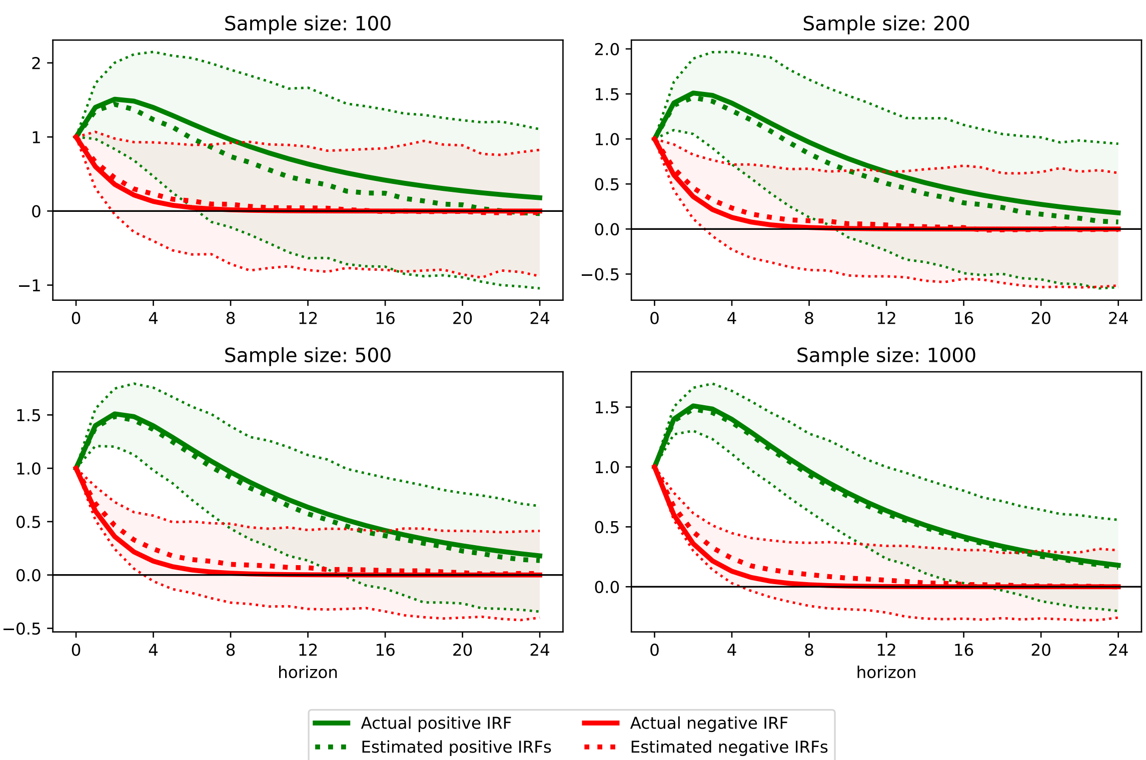


Note: IRFs of a $AR(2)$ process with parameters $\phi_1^+ = 1.4$, $\phi_2^+ = -0.45$ and a $AR(1)$ process with parameter $\phi_1^- = 0.6$.

Figure 2: The IRFs of positive and negative shocks of the generated series

Next, I estimate the impulse response functions of positive and negative shocks with the proposed 2-step local projections methodology.

I plot the estimated IRFs along with the actual IRFs in Fig. 3.



Note: Number of samples = 1000.
Dashed lines represent 2.5-th, 50-th and 97.5-th percentiles of estimated IRFs

Figure 3: Estimated versus actual IRFs

5.2 Estimation of asVMA processes

Next, I extend the analysis to multivariate $asMA$ processes. I generate a bivariate $asMA$ process with the following features. The parameters corresponding to positive and negative innovations are given by the VMA representations of different $VAR(1)$ process with the following parameters

$$\boldsymbol{\Phi}^{+}=\begin{bmatrix}1.05 & -0.1\\ 0.2 & 0.8\end{bmatrix} \qquad \boldsymbol{\Phi}^{-}=\begin{bmatrix}0.6 & 0.2\\ 0.1 & 0.8\end{bmatrix} \qquad \boldsymbol{A}_0^{-1}=\begin{bmatrix}1 & 0\\ 0.5 & 1\end{bmatrix}$$

The matrix $\boldsymbol{A}_0^{-1}$ is lower triangular and implies contemporaneous exogeneity of the first variable.

Then, I draw 1000 samples of the bivariate $asVMA$ process with sample size of 500 and estimate the impulse response functions of positive and negative shocks with the proposed 2-step local projections methodology. The identification of the structural shocks is done by recursive ordering in line with the assumptions of the data generating process.

I plot the estimated IRFs along with the actual IRFs in Fig. 4.

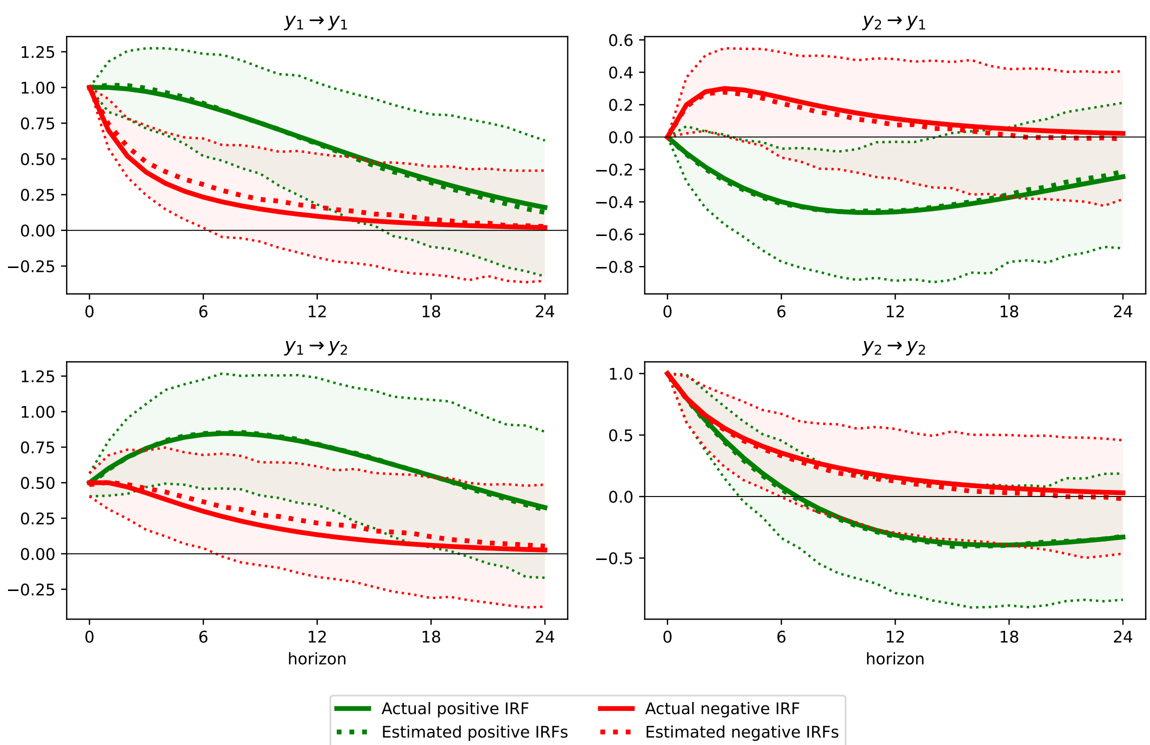


Figure 4: Estimated versus actual IRFs

Note: Number of samples = 1000.
Dashed lines represent 2.5-th, 50-th and 97.5-th percentiles of estimated IRFs

The results show that the proposed methodology has satisfactory performance in estimating the impulse response functions of positive and negative shocks in both univariate and multivariate $asMA$ processes.

Conclusions

This paper develops a methodology for estimating asymmetric moving average processes in both univariate and multivariate settings. The proposed methodology is based on the Wold representation of the process and the use of proxies for the structural shocks. The methodology is simple, computationally efficient, and can be implemented with standard statistical software. The Monte Carlo analysis shows that the proposed methodology performs well in both finite and infinite samples.

References

Barnichon, R. and Matthes, C. (2018). Functional Approximation of Impulse Responses. *Journal of Monetary Economics*, 99:41–55.

Brännäs, K. and Gooijer, J. G. D. (2004). Asymmetries in conditional mean and variance: modelling stock returns by asMA-asQGARCH. *Journal of Forecasting*, 23(3):155–171.

Brännäs, K., Gooijer, J. G. D., Lönnbark, C., and Soultanaeva, A. (2012). Simultaneity and Asymmetry of Returns and Volatilities: The Emerging Baltic States' Stock Exchanges. *Studies in Nonlinear Dynamics & Econometrics*, 16(1).

Brånnås, K. and De Gooijer, J. G. (1994). Autoregressive-asymmetric moving average models for business cycle data. *Journal of Forecasting*, 13(6):529–544.

De Gooijer, J. G. (2021). Asymmetric vector moving average models: estimation and testing. *Computational Statistics*, 36(2):1437–1460.

Jordà, Ò. (2005). Estimation and Inference of Impulse Responses by Local Projections. *American Economic Review*, 95(1):161–182.

Wecker, W. E. (1981). Asymmetric Time Series. *Journal of the American Statistical Association*, 76(373):16–21.